Slepian Scale-Discretised Wavelets on the Sphere arxiv:2106.02023

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Introduction

Problem Often data are not observed over the whole sphere and are missing in some region. Wavelets allow one to probe spatially localised, scale-dependent features of signals on the sphere. However, the boundaries of the region of missing data contaminate nearby wavelet coefficients.

Solution A possible approach to solve this problem is to construct wavelets within the region itself.

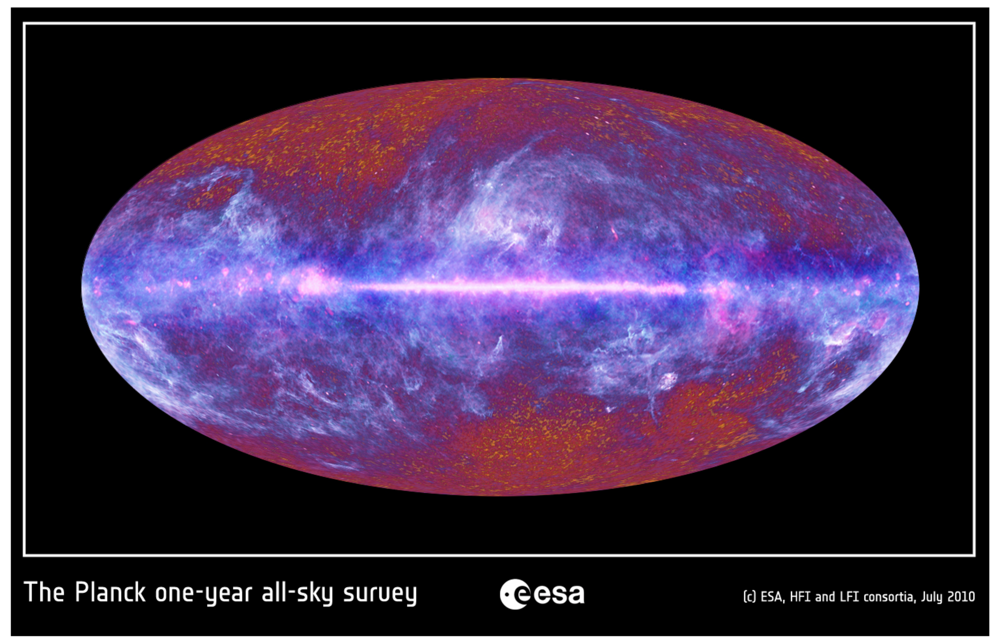

Figure 1: In cosmic microwave background analyses the region around the Galactic plane is often removed [1].

Slepian Concentration Problem

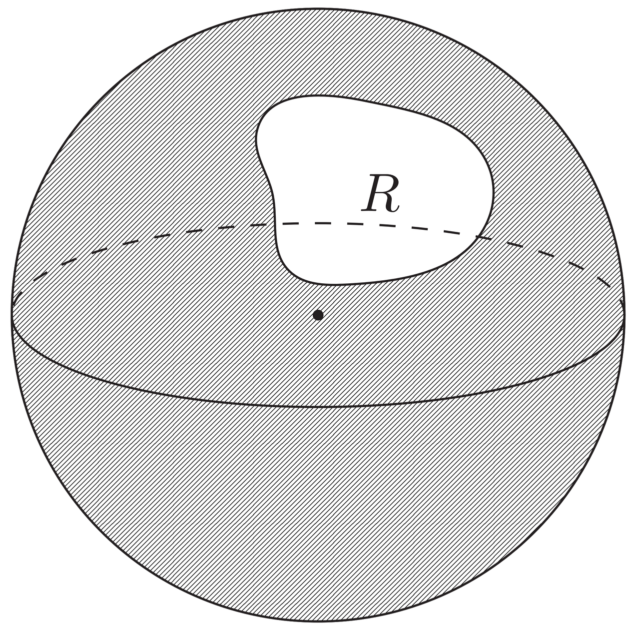

Figure 2: Often data are observed on a partial region of the sphere only.

A function cannot be strictly spacelimited as well as strictly bandlimited [2, 3]. The Slepian functions S_p are optimally concentrated within a region R. To maximise the spatial concentration of a bandlimited function $f \in L^2(\mathbb{S}^2)$ within a region R one must maximise the following ratio:

$$\mu = \frac{\int_R \mathrm{d}\Omega(\omega)\,|f(\omega)|^2}{\int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega)\,|f(\omega)|^2}, \tag{1}$$

where $0 < \mu < 1$ is a measure of the spatial concentration [4]. A bandlimited function f can be decomposed into this basis

$$f(\omega) = \sum_{p=1}^{L^2} f_p S_p(\omega). \tag{2}$$

Wavelets Construction

Sifting Convolution The *sifting convolution* [5] (developed by the authors of this poster) can be extended to any arbitrary basis. The translation of an arbitrary function f is

$$(\mathcal{T}_{\omega'} f)(\omega) = \sum_p f_p S_p(\omega') S_p(\omega). \tag{3}$$

The sifting convolution between two functions f, g is

$$(f \circledcirc g)(\omega) = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega')\,(\mathcal{T}_\omega f)(\omega') g^*(\omega'), \tag{4}$$

which is a product in Slepian space

$$(f \circledcirc g)_p = f_p g_p^*. \tag{5}$$

Slepian Wavelets Wavelet coefficients W^{Ψ^j} may be defined by a sifting convolution of f with the wavelet Ψ^j for wavelet scale j:

$$W^{\Psi^j}(\omega) = (\Psi^j \circledcirc f)(\omega). \tag{6}$$

Similarly, scaling coefficients W^Φ are defined by a convolution between f and the scaling function Φ:

$$W^\Phi(\omega) = (\Phi \circledcirc f)(\omega). \tag{7}$$

The function f may be reconstructed from its wavelet and scaling coefficients by

$$f(\omega) = (\Phi \circledcirc W^\Phi)(\omega) + \sum_{j=J_0}^{J} (\Psi^j \circledcirc W^{\Psi^j})(\omega). \tag{8}$$

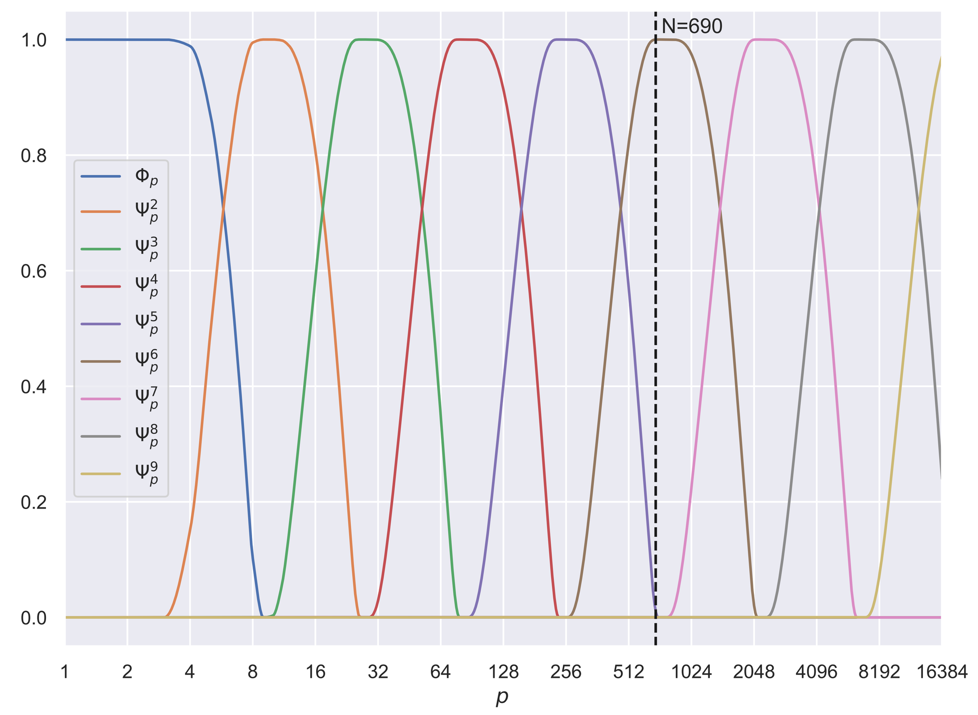

Figure 3: The Slepian wavelets are constructed by a tiling of the Slepian line.

Numerical Illustration

A region on the sphere is constructed from the *Earth Gravitational Model EGM2008* dataset [?].

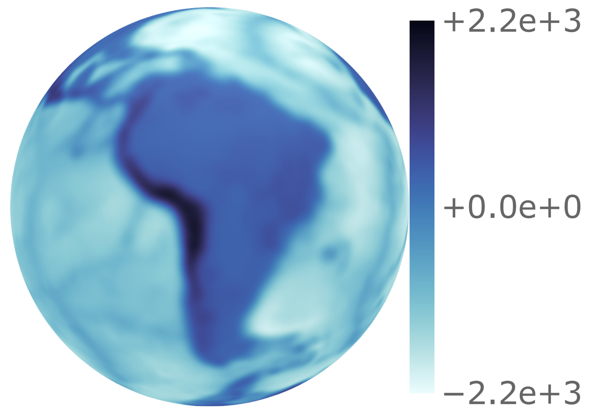

Figure 4: EGM2008

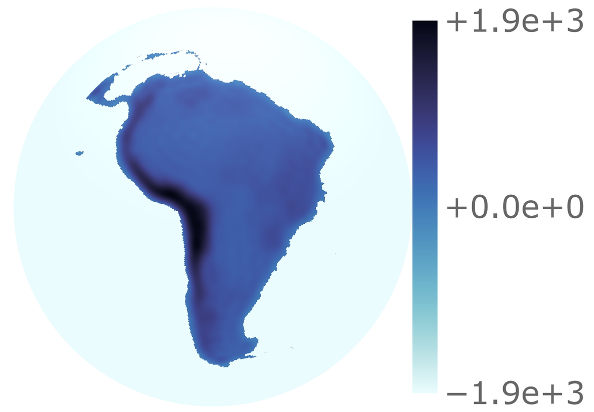

Figure 5: R

The Slepian functions are less-well concentrated for higher p.

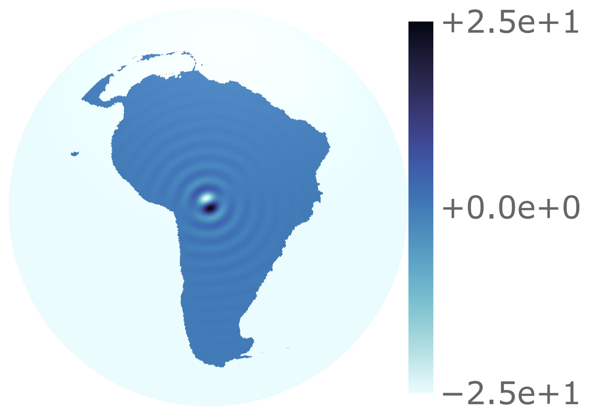

Figure 6: $S_1(\omega),\ \mu = 1.00$

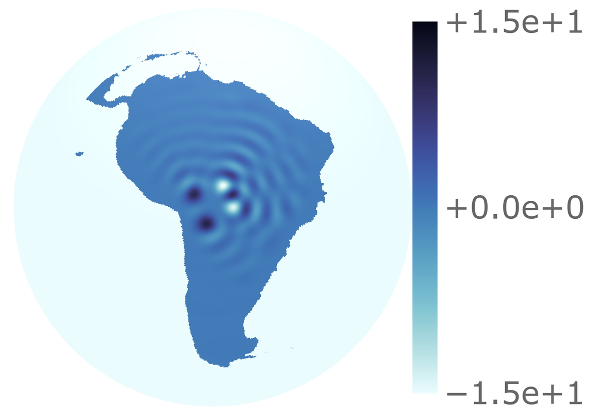

Figure 7: $S_{10}(\omega),\ \mu = 1.00$

The scaling function and first wavelet.

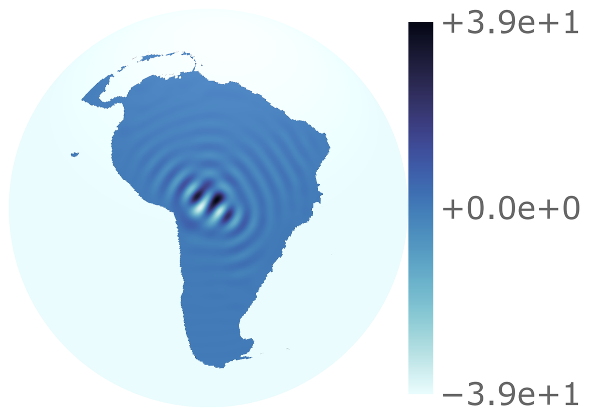

Figure 8: $\Phi(\omega)$

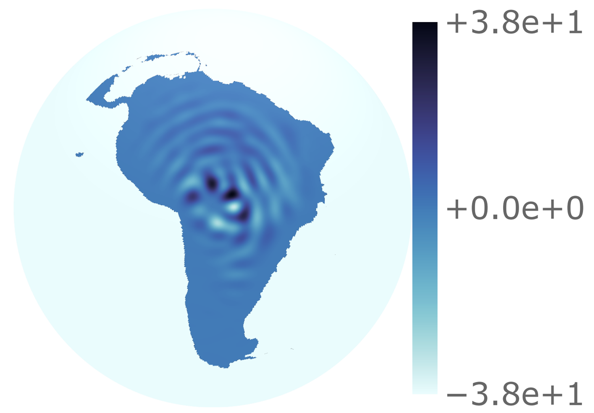

Figure 9: $\Psi^{2j}(\omega)$

The scaling coefficient and the first wavelet coefficient.

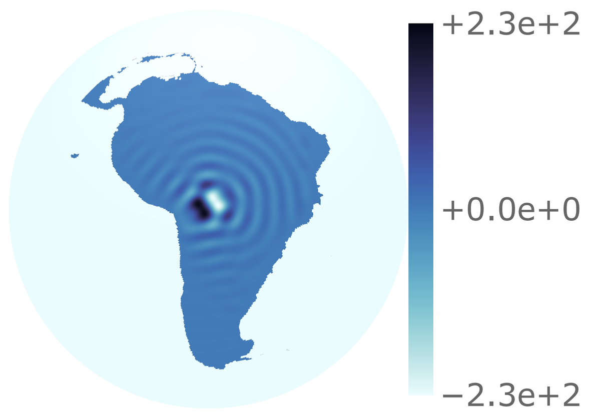

Figure 10: $W^\Phi(\omega)$

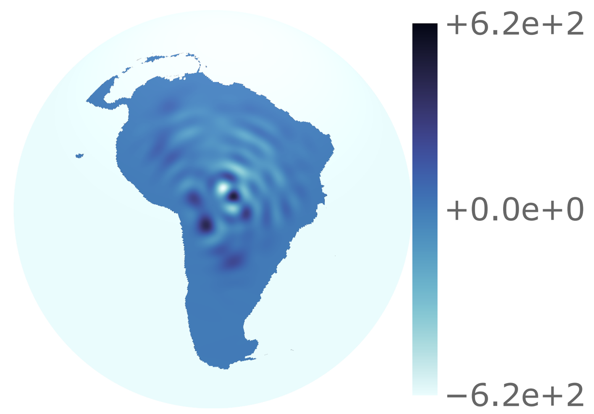

Figure 11: $W^{\Psi^{2j}}(\omega)$

Denoising Example

Consider a signal localised in R in the presence of noise

$$x(\omega) = s(\omega) + n(\omega). \tag{9}$$

Homogeneous, isotropic white noise is defined as

$$\langle n_{\ell m} n_{\ell' m'}^* \rangle = \sigma^2 \delta_{\ell\ell'} \delta_{mm'}, \tag{10}$$

which defines the noise in Slepian space:

$$\langle n_p n_{p'}^* \rangle = \sigma^2 \delta_{pp'}. \tag{11}$$

The denoised wavelet coefficients $D^\varphi(\omega) = (\varphi \circledcirc d)(\omega)$, where $\varphi \in \{\Phi, \Psi^j\}$, become

$$D^\varphi(\omega) = \begin{cases} 0, & X^\varphi(\omega) < N_\sigma \sigma^\varphi(\omega), \\ X^\varphi(\omega), & X^\varphi(\omega) \geq N_\sigma \sigma^\varphi(\omega). \end{cases} \tag{12}$$

A clear boost in signal-to-noise is observed.

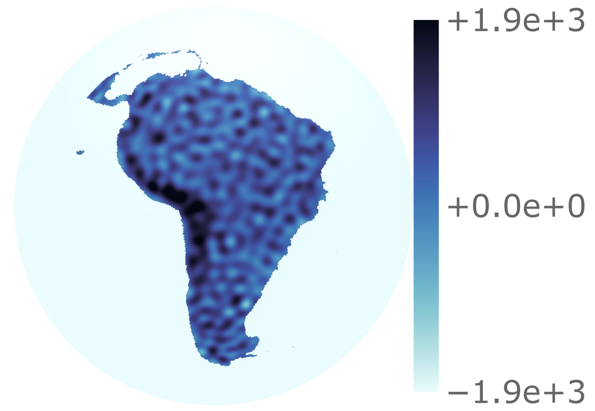

Figure 12: Noisy data
$\mathrm{SNR}(x) = 4.11$ dB

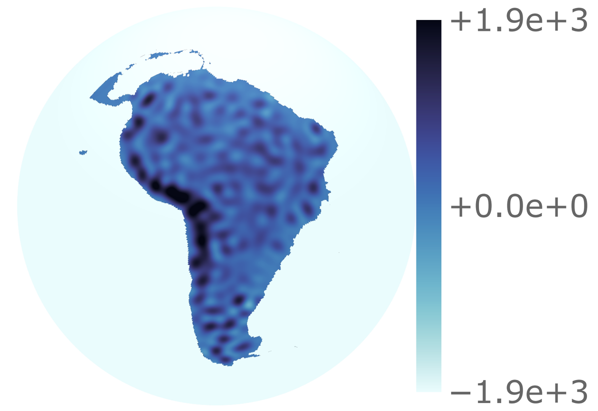

Figure 13: $N_\sigma = 2$
$\mathrm{SNR}(d) = 5.67$ dB

Properties

- In contrast to the spherical harmonic setting, low and high p represent high and low concentration respectively
- Wavelet energy $\|\varphi\|^2 = \sum_p |\varphi_p|^2$
- Wavelets satisfy a *Parseval frame*
- Wavelet variance depends on the position on the sphere $[\Delta W^\varphi(\omega)]^2 = \sum_p \sigma^2 |\varphi_p|^2 |S_p(\omega)|^2$

References

[1] D. J. Mortlock, A. D. Challinor, and M. P. Hobson, *Mon. Not. R. Astron. Soc.*, vol. 330, no. 2, pp. 405–420, 2002.

[2] D. Slepian and H. O. Pollak, *Bell Syst. Tech. J.*, vol. 40, no. 1, pp. 43–63, 1961.

[3] D. Slepian, *SIAM Rev.*, vol. 25, no. 3, pp. 379–393, 1983.

[4] F. J. Simons, F. A. Dahlen, and M. A. Wieczorek, *Soc. Ind. Appl. Math.*, vol. 48, no. 3, pp. 504–536, 2006.

[5] P. J. Roddy and J. Mcewen, *IEEE Signal Process. Lett.*, 2021.